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SHORT COMMUNICATION

NOISE PROBLEMS WITH EPHEMERAL STREAMS IN AQUIFER MODELS

B. J. CONNORTON
Thames Water Authority
AND
W. L. WOOD
Department of Mathematics, University of Reading, Reading RG6 2AX England

SUMMARY

Previous papers[1,2] have drawn attention to the sustained oscillations ('noise') in the solution by successive over-relaxation of the equations from the finite difference approximation of regional groundwater flow including ephemeral streams. This paper shows that the trouble can be avoided by introducing an averaging step in the algorithm; the trouble can also be avoided by 'under-relaxation' but this is far less efficient than averaging.

KEY WORDS Aquifer Model Leakage Noise Averaging

INTRODUCTION

Connorton and Hanson[1] and Rushton and Tomlinson[2] have drawn attention to computational difficulties encountered when incorporating an ephemeral stream system into a numerical model of regional groundwater flow. The conventional partial differential equation used by these authors as the basis for the mathematical model of the groundwater flow is as given by Bear:[3]

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h_A}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y \frac{\partial h_A}{\partial y}\right)=S\frac{\partial h}{\partial t}-q \tag{1}$$

where x, y are horizontal Cartesian co-ordinates.

$h_A = h_A(x, y, t)$ is the average piezometric head

T_x, $T_y = T_x(x, y)$, $T_y(x, y)$ are the transmissivities in the x and y directions respectively.

$S = S(x, y)$ is the storativity

$q = q(x, y, h_A, t)$ is the sink or source term.

Included within the term q is not only pumpage and natural or artificial recharge but also ephemeral stream leakage. It is the latter which makes this term dependent on the piezometric head h_A and hence dependent on the solution.

The boundary conditions are such that at every point on the boundary of the region we have either a Dirichlet boundary condition with given value of h_A or a no-flow Neumann condition.

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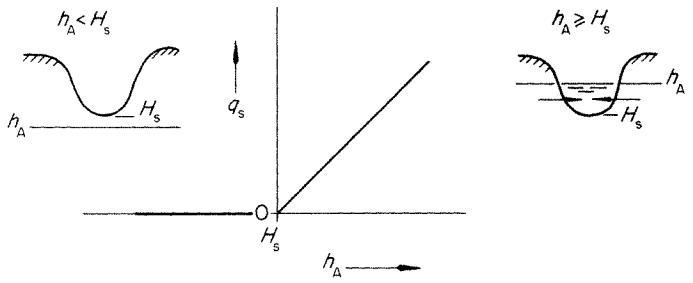

Figure 1. Graphical representation of stream leakage

Equation (1) is solved numerically by Connorton and Hanson[1] and Rushton and Tomlinson[2] using finite differences and successive over-relaxation.[4] Clearly the iteration necessary because of the ephemeral streams could also be combined with solving the simultaneous equations directly. In both the first two references[1,2] the stream leakage is of the form

$$q_s = K_s(x, y) f(\Delta h_A) \tag{2}$$

where $K_s(x, y)$ is the stream leakage parameter and $f(\Delta h_A)$ is some function of $\Delta h_A = h_A(x_s, y_s, t) - H_s(x_s, y_s)$ where (x_s, y_s) is a point on the stream system and H_s is some fixed datum associated with the stream, usually the stream bed level.

The particular case investigated here is when $f(\Delta h_A)$ is a linear function such that

$$q_s = \begin{cases} K_s(h_A - H_s), & h_A \geq H_s \\ 0, & h_A < H_s \end{cases} \tag{3}$$

Equations (3) imply that the stream has no storage capacity; when $h_A < H_s$ the stream dries up and no leakage from the stream into the aquifer can take place, see Figure 1.

In the Connortion and Hanson paper[1] the computational difficulties took the form of sustained oscillations ('noise') such that the iteration never converged, as shown in Figure 2, which shows the head h_A and q_s at a specific stream node on the finite difference mesh at successive iterations.

The analysis of the trouble depends on the exact algorithm used. This paper analyses an algorithm for the solution of the problem with leakage from an ephemeral stream as given by equation (3) which can produce the sustained oscillation and shows that the trouble can be avoided by the addition of one extra step in the algorithm.

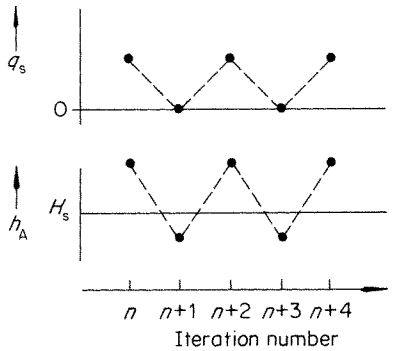

Figure 2. Sustained oscillation at a stream node

The key non-dimensional parameter is

$$B = \frac{K_s \Delta x b}{4T} \tag{3a}$$

where T is the local average transmissivity, Δx is the finite difference mesh size ($\Delta x = \Delta y$ for simplicity here) and b is the local average width of the stream. Experience has shown that the larger the value of B the greater is the likelihood of noise oscillation troubles with the original algorithm. For typical regional flow problems the areal dimensions of the aquifer may be such that it is not possible to reduce B by reducing the mesh size Δx and we must necessarily have Δx large compared with the width of the stream b. For the Lambourn aquifer, for example,[1] B can have values up to 10^4.

THE NUMERICAL SOLUTION

We consider the steady state equation without sources or sinks since this is sufficient to illustrate the oscillation troubles and the remedy:

$$\frac{\partial}{\partial x}\left(T\frac{\partial h_A}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial h_A}{\partial y}\right)=[K_s(h_A-H_s)] \tag{4}$$

where the right-hand term only applies for the part of the region occupied by an ephemeral stream with $h_A \geq H_s$.

We put $h_A = hH_s$, supposing H_s is constant for simplicity. Then, using Varga's integration method[4] with $\Delta x = \Delta y$ (i.e. square mesh) we have for the numerical solution typical equations

$$-\frac{1}{4T}T_{i,j-\frac{1}{2}}h_{i,j-1}-\frac{1}{4T}T_{i-\frac{1}{2},j}h_{i-1,j}+h_{i,j}-\frac{1}{4T}T_{i+\frac{1}{2},j}h_{i+1,j} -\frac{1}{4T}T_{i,j+\frac{1}{2}}h_{i,j+1}+\left[K_s\frac{\Delta x b}{4T}h_{i,j}\right]=\left[K_s\frac{\Delta x b}{4T}\right] \tag{5}$$

where

$$T_{i,j} = T(i\Delta x, j\Delta y)$$
$$h_{i,j} = h(i\Delta x, j\Delta y)$$

and

$$4T = T_{i,j-\frac{1}{2}}+T_{i-\frac{1}{2},j}+T_{i,j+\frac{1}{2}}+T_{i+\frac{1}{2},j}$$

We suppose $b < \Delta x$ and for simplicity that the stream runs along a mesh line. The extra terms in square brackets in equation (5) are only present when $(i\,\Delta x, j\,\Delta y)$ is a stream node. Put

$$B = \frac{K_s \Delta x b}{4T}$$

Suppose the nodes are re-numbered as a one-dimensional array $\{l\}$ and that the matrix from the finite difference operator corresponding to the non-bracketed terms in equation (5) has entries $a_{l,m}$. Then the S.O.R. algorithm as originally applied to the set of equations (5) is given by

$$h_l^{k+1} = (1-\omega)h_l^k + \frac{\omega}{1+B}\left[f_l + B - \sum_{m<l} a_{l,m}h_m^{k+1} - \sum_{m>l} a_{l,m}h_m^k\right] \tag{6}$$

if $h_l^k \geq 1$ and l is the number of a stream node or

$$h_l^{k+1} = (1-\omega)h_l^k + \omega\left[f_l - \sum_{m<l} a_{l,m}h_m^{k+1} - \sum_{m>l} a_{l,m}h_m^k\right] \tag{7}$$

otherwise

(where ω is the S.O.R. parameter and f_l comes from any Dirichlet boundary data affecting the equation).

The superscript k in equations (6) and (7) denotes the number of the iteration. The diagonal terms $a_{l,l}$ are each unity.

The initial values h_l^0 are taken below the stream bed levels i.e. $h_l^0 < 1$, for all l, and the iteration proceeds without the extra terms in the equations and with the values h_l^k increasing until we have $h_s^k \geq 1$ for some stream node s. Then if $\hat{h}_s^{k+1}$ is the value we would get if the next step did *not* include the extra terms, this is given by

$$\hat{h}_s^{k+1} = (1-\omega)h_s^k + \omega\left[f_s - \sum_{m<s} a_{s,m}h_m^{k+1} - \sum_{m>s} a_{s,m}h_m^k\right] \tag{8}$$

However, what we actually compute is

$$h_s^{k+1} = \frac{B(1-\omega)}{1+B}h_s^k + \frac{1}{1+B}\hat{h}_s^{k+1} + \frac{\omega B}{1+B} \tag{9}$$

Now we know that $h_s^k = 1+\varepsilon_s^k$, $\varepsilon_s^k \geq 0$. Suppose $\hat{h}_s^{k+1} = 1+\hat{\varepsilon}_s^{k+1}$. Substituting in equation (9) we have

$$h_s^{k+1} = 1 - \frac{(\omega-1)B}{1+B}\varepsilon_s^k + \frac{\hat{\varepsilon}_s^{k+1}}{1+B} \tag{10}$$

Hence

$$h_s^{k+1} < 1 \quad \text{if} \quad (\omega-1)B\varepsilon_s^k > \hat{\varepsilon}_s^{k+1} \tag{11}$$

The S.O.R. parameter ω was taken as 1·6 for the Lambourn aquifer model,[1] and the value of B could be up to 10^4, hence condition (11) is very likely to apply. The consequence is that a persistent oscillation may be set up. We can have $h_s^{k-1} < 1$, $h_s^k > 1$, $h_s^{k+1} < 1$, etc. with extra terms brought in at alternate sweeps and the process will never converge.

The cure is to insert another step in the algorithm so that we have

$$\hat{h}_l^k = \tfrac{1}{2}(h_l^k + \hat{h}_l^{k-1}) \tag{12}$$

followed by

$$h_l^{k+1} = (1-\omega)\hat{h}_l^k + \frac{\omega}{1+B}\left[f_l - \sum_{l<m} a_{l,m}h_m^{k+1} - \sum_{l>m} a_{l,m}\hat{h}_m^k + B\right] \tag{13}$$

if $\hat{h}_l^k \geq 1$ and l is a stream node, or

$$h_l^{k+1} = (1-\omega)\hat{h}_l^k + \omega\left[f_l - \sum_{l<m} a_{l,m}h_m^{k+1} - \sum_{l>m} a_{l,m}\hat{h}_m^k\right] \tag{14}$$

otherwise.

If $\mathbf{L}$ is the S.O.R. iteration matrix corresponding to the set of equations (7) and $\bar{\mathbf{h}}$ is the limit to which this iteration would converge, we have

$$\mathbf{h}^{k+1} - \bar{\mathbf{h}} = \mathbf{L}(\hat{\mathbf{h}}^k - \bar{\mathbf{h}})$$

Hence, from equation (12)

$$\hat{\mathbf{h}}^{k+1}-\bar{\mathbf{h}}=\tfrac{1}{2}[\mathbf{I}+\mathbf{L}][\hat{\mathbf{h}}^{k}-\bar{\mathbf{h}}] \tag{15}$$

The iteration is thus still convergent, because if λ is an eigenvalue of $\mathbf{L}$, in the new iteration matrix the eigenvalues are $\frac{1}{2}(1+\lambda)$ and we known that $0<\frac{1}{2}(1+\lambda)<1$.

Now again the iteration proceeds with the values of $\hat{h}_s$ increasing until we reach $\hat{h}_s^k=1+\hat{\varepsilon}_s^k$, $\hat{\varepsilon}_s^k\geq 0$. Again suppose $\hat{h}_s^{k+1}=1+\hat{\varepsilon}_s^{k+1}$ is the value at this stream node that would be given without the extra terms, then what we actually compute is

$$h_s^{k+1}=\frac{(1-\omega)B}{1+B}\hat{h}_s^k+\frac{1}{1+B}\hat{h}_s^{k+1}+\frac{\omega B}{1+B} \tag{16}$$

Hence from equation (12)

$$\hat{h}_s^{k+1}=1+\tfrac{1}{2}\frac{[1+B(2-\omega)]}{1+B}\hat{\varepsilon}_s^k+\frac{1}{2(1+B)}\hat{\varepsilon}_s^{k+1} \tag{17}$$

The situation now is that even if $\hat{\varepsilon}_s^{k+1}$ is negative the relatively large factor attached to the $\hat{\varepsilon}_s^k$ term is positive. There may be some small oscillation early in the procedure but it always eventually settles down so that $\hat{h}_s^k>1$ is followed by $\hat{h}_s^{k+1}>1$ also and the iteration converges to the solution of the equations with the extra terms as appropriate.

It is apparent from equation (10) that the oscillation trouble could also be avoided by making $\omega<1$, i.e. by under-relaxation, but this gives much slower convergence than the averaging method. We can demonstrate why this should be so by reference to the standard S.O.R. theory as in Reference 4. The formula for $\bar{\omega}$, the optimum value of the S.O.R. parameter is

$$\bar{\omega}=\frac{2}{1+(1-\mu^2)^{1/2}} \tag{18}$$

where μ is the spectral radius of the corresponding Jacobi matrix. The dominant eigenvalue of the S.O.R. iteration matrix is then $\lambda_1=\bar{\omega}-1$.

The effect of averaging is to make the dominant eigenvalue

$$\lambda_a=\tfrac{1}{2}(1+\lambda_1)=\frac{\bar{\omega}}{2} \tag{19}$$

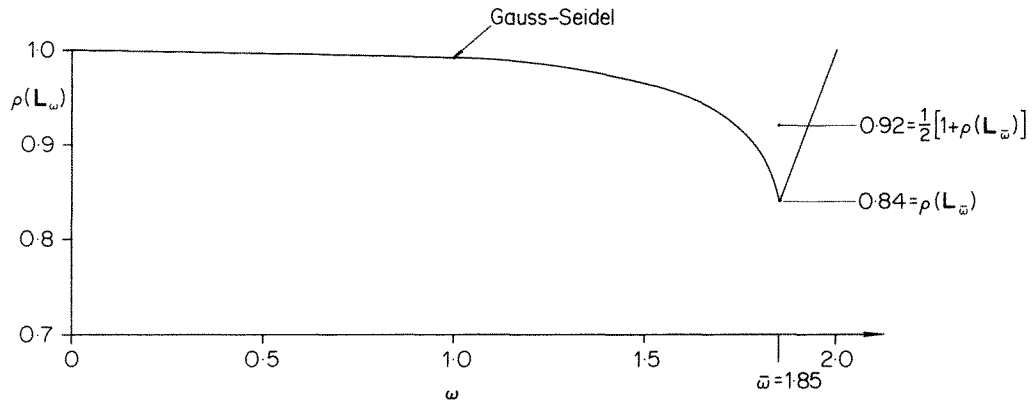

Figure 3. Graph of spectral radius of S.O.R. iteration matrix against ω.

If this gives faster convergence than the Gauss–Seidel method with $\omega = 1$ then it certainly gives faster convergence than with $\omega < 1$ as can be seen from the graph of the spectral radius of the iteration matrix against ω. (Figure 3 shows this graph as it applies to the numerical model of the Cotswold limestone aquifer referred to in the next section). The spectral radius of the Gauss–Seidel iteration matrix is μ^2 (Reference 4). Hence averaging gives faster convergence than under-relaxation if

$$\lambda_a < \mu^2 \tag{20}$$

i.e.

$$\frac{1}{1+(1-\mu^2)^{1/2}} < \mu^2 \tag{21}$$

from equations (18) and (19).

Inequality (21) implies $\mu > 0{\cdot}780$ approximately. Since $\cos(\pi/5) = 0{\cdot}8090$, which is the estimate for μ for Laplace on a square with 5×5 mesh and Dirichlet boundary condition, inequality (20) is certainly likely to be satisfied for a practical aquifer model as demonstrated in the next section.

APPLICATION TO NUMERICAL MODEL

The averaging procedure described above was applied to a finite difference model (based on equation (1)) of a Cotswold limestone aquifer (see Figure 4). The model is sub-divided into

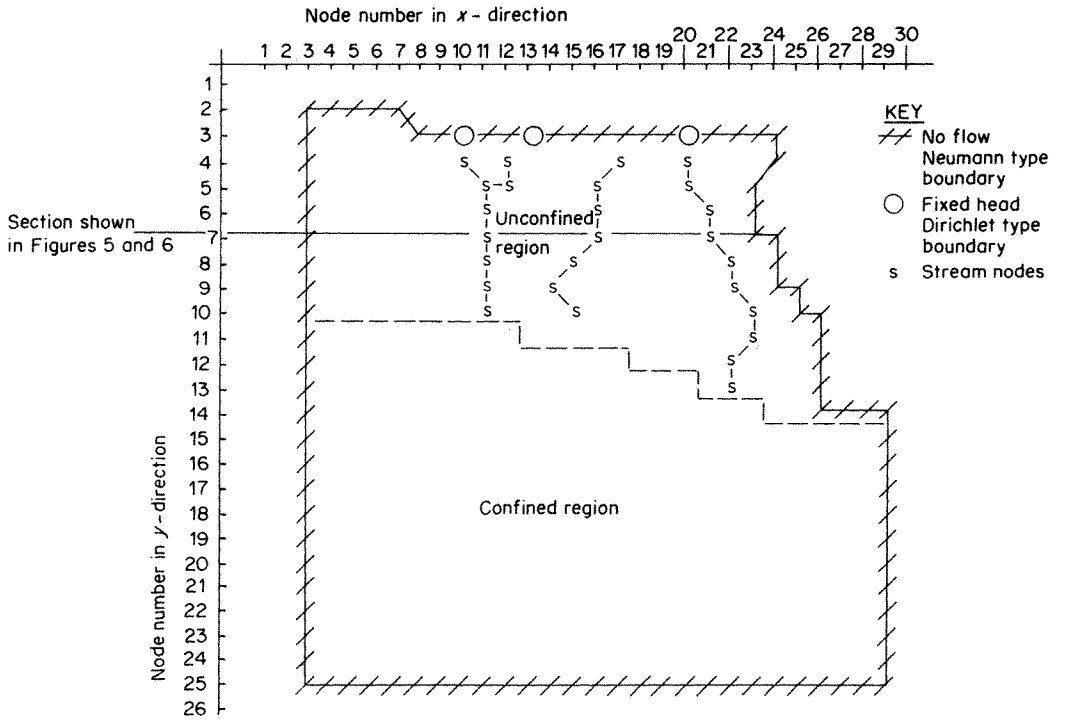

Figure 4. Finite difference grid for Cotswold Limestone aquifer

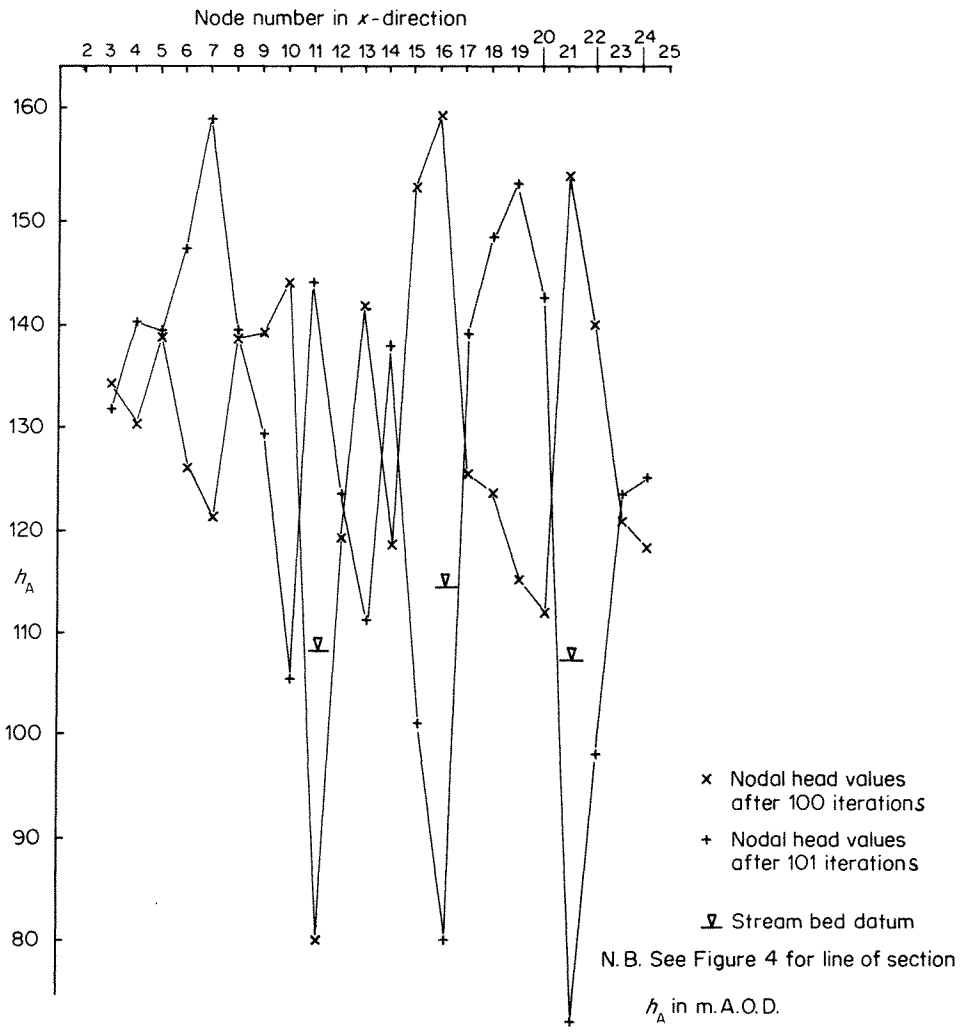

Figure 5. Example of the oscillation in nodal head values from row 7 of the Costwold Limestone aquifer model at the 100 and 101 iterations

an unconfined and a confined region. Three ephemeral streams, whose flow mechanism is of the leakage type defined by equation (3) above, flow over the unconfined region in the direction of the confined region.

Initially, transmissivity (T_x, T_y) for the unconfined region was set to 500 m^2/d and satisfactory results were obtained using the straightforward successive over-relaxation iteration defined above by equations (6) and (7). However, when transmissivity was reduced to 100 m^2/d thereby increasing the parameter B (equation (3a)), sustained oscillations of the type described by Connorton and Hanson[1] were obtained. This spurious behaviour is demonstrated in Figure 5 which shows a section along row 7 of nodal head values taken at the 100 and 101 iterations. The sustained oscillations occurred globally but, as might be expected, were most pronounced in the vicinity of the streams; the oscillations had not stopped after 900 iterations.

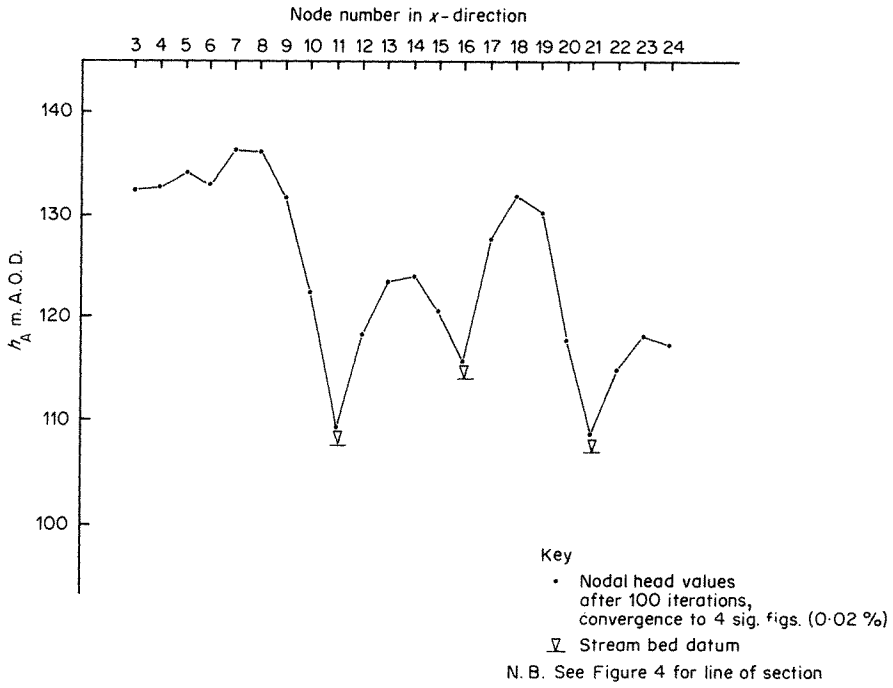

Figure 6. Converged solution of the nodal head values from row 7 of the Cotswold Limestone aquifer model after 100 iterations using averaging

Applying the averaging procedure defined by equation (12) in conjunction with an optimized value for ω of 1·85 gave convergence to four significant figures after 100 iterations (see Figure 6).

Using various values of $\omega < 1$ without averaging also gave convergent results. However, the rate of convergence for under-relaxation alone was found to be appreciably slower than that for the averaging algorithm used in conjunction with an optimized over-relaxation parameter, as indicated by the above theory.

ACKNOWLEDGEMENT

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REFERENCES

1. B. Connorton and C. A. Hanson, 'Regional modelling—analogue and digital approaches' *Thames Groundwater Scheme Conference Proceedings*, April 1978.
2. K. R. Rushton and L. M. Tomlinson, 'Possible mechanisms for leakage between aquifers and rivers', Dept. of Civil Engineering, University of Birmingham, March 1978.
3. J. Bear, *Hydraulics of Groundwater*, McGraw Hill, 1979.
4. R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, 1962.